Recovery of Graded Index Profile of Planar Waveguide by Cubic Spline Function *

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A method is proposed to recover the refractive index profile of graded waveguide from the effective indices by a cubic spline interpolation function. Numerical analysis of several typical index distributions show that the refractive index profile can be reconstructed closely to its exact profile by the presented interpolation model.

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The determination of the refractive index profile plays a fundamental role in a graded index waveguide because it can give significant information of waveguide propagation properties, thus it has attracted considerable interests in past decades. The index recover techniques used to solve this problem can be classified into the destructive and non-destructive methods. In the destructive method, samples must be processed before the measurement, such as reflectivity profiling[1] and ellipsometry.[2] The non-destructive method takes advantage of intact of waveguide sample. The commonly used method such as inverse Wentzel–Kramers–Brillouin (IWKB) methods[3−7] is employed to reconstruct index profile from measured effective indices. However, the Wentzel–Kramers–Brillouin (WKB) analysis is an approximate method, and it has a low accuracy for its inverse methods. With regarding to the waveguides supporting fewer modes, additional techniques such as changing the cover layer should be adopted, making the recovery process complicated. In this Letter, an interpolation method with cubic spline function[8] based on an exact analytic transfer matrix (ATM) method[9,12] is introduced to recover the refractive index profile from effective indices. The cubic spline has a quite smooth profile, it ensures that the first and second derivatives at each interpolation point are continuous, which matches the situation of real index profile. With an iterative procedure, the index profile can be recovered by this method with a good accuracy.

A series of mode indices n_i $(i = 0, 1, \cdots, k)$ of graded-index waveguide can be measured by m-line method.[10] A set of increasing coordinate x_i $(i = 0, 1, \cdots, k)$ is assumed arbitrarily and assigned to mode index $\{n_i\}$. The surface index is n_c $(x_c = 0)$ is unknown and should be determined later. Now we have a sets of points $[(n_c, 0), (n_0, x_0), (n_1, x_1), \cdots, (n_k, x_k)]$ and these points can be interpolated by cubic spline functions with a smooth profile. In order to work out the unknown surface index n_c, the point (n_0, x_0) is picked out from the sets of points. Thus the series of points $[(n_c, 0), (n_1, x_1), (n_2, x_2), \cdots, (n_k, x_k)]$ except point (n_0, x_0) are finally used to fit the cubic spline. When $x \geq x_k$, an exponential profile is assumed. The index profile can be described as follows:

$$n(x) = \begin{cases} M_0\dfrac{(x_1-x)^3}{6h_0} + M_1\dfrac{x^3}{6h_0} \\ \quad +\Big(n_c - \dfrac{M_0h_0^2}{6}\Big)\dfrac{x_1-x}{h_0} \\ \quad +\Big(n_1 - \dfrac{M_1h_0^2}{6}\Big)\dfrac{x}{h_0}, \quad 0 \leq x \leq x_1, \\ M_i\dfrac{(x_{i+1}-x)^3}{6h_i} + M_{i+1}\dfrac{(x-x_i)^3}{6h_i} \\ \quad +\Big(n_i - \dfrac{M_ih_i^2}{6}\Big)\dfrac{x_{i+1}-x}{h_i} \\ \quad +\Big(n_{i+1} - \dfrac{M_{i+1}h_i^2}{6}\Big)\dfrac{x-x_i}{h_i}, \\ \quad x_i \leq x \leq x_{i+1},\ i = 1, 2, \cdots, k-1, \\ n_s + b\cdot\exp(-ax), \qquad x \geq x_k, \end{cases} \tag{1}$$

where $h_0 = x_1$, $h_i = x_{i+1} - x_i$, $(i = 1, 2, \cdots, k-1)$; M_i $(i = 0, 1, \cdots, k)$ is the second derivative of $n(x)$ at the sets of points $[(n_c, 0), (n_1, x_1), (n_2, x_2), \cdots, (n_k, x_k)]$; n_s is the refractive index of the substrate. Parameters a and b of the exponential profile beyond x_k are determined by two points $[(n_k, x_k), (n_{k+1}, x_{k+1})]$, where $x_{k+1} \gg x_k$ and $n_{k+1} = n_s + \varepsilon$; ε is very small value. It is proven by calculation that $x_{k+1} = 5x_k$ to $10x_k$ and $\varepsilon = 1\times10^{-10}$ to 1×10^{-7} have little influence on the calculation results. In the simulation, we set $x_{k+1} = 5x_k$ and $\varepsilon = 1\times10^{-7}$. The series of second derivatives M_i $(i = 0, 1, \cdots, k)$ are determined by the following system of linear equations:

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$$\begin{bmatrix} 2 & 1 & & & \\ \mu_1 & 2 & \gamma_1 & & \\ & \vdots & \vdots & \vdots & \\ & & \mu_{k-1} & 2 & \gamma_{k-1} \\ & & & 1 & 2 \end{bmatrix} \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{k-1} \\ M_k \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{k-1} \\ d_k \end{pmatrix}, \tag{2}$$

where $\mu_i = \dfrac{h_{i-1}}{h_{i-1}+h_i}$, $\gamma_i = 1-\mu_i$, $(i = 1, 2, \cdots, k-1)$, $d_0 = \dfrac{6}{h_0}\Big[\dfrac{n_1 - n_c}{h_0} - n'(0)\Big]$, $n](0)$ is the first derivative of $n(x)$ at the surface $x = 0$; $n'(0)$ will be artificially given and may be changed, and the method to close its value will be discussed in detail in the following. The expressions of d_k $(k = 0, 1 \cdots k)$ are

$$d_1 = 6\frac{\dfrac{n_1 - n_c}{h_0} - \dfrac{n_2 - n_1}{h_1}}{-x_2}, \tag{3}$$

$$d_i = 6\frac{\dfrac{n_1 - n_{i-1}}{h_{i-1}} - \dfrac{n_{i+1} - n_i}{h_i}}{x_{i-1} - x_{i+1}}, \quad i = 2, \cdots, k-1, \tag{4}$$

$$d_k = \frac{6}{h_{k-1}}\Big[n'(x_k) - \frac{n_k - n_{k-1}}{h_{k-1}}\Big], \tag{5}$$

where $n'(x_k)$ is the first derivative of $n(x)$ at $x = x_k$, which is equal to the first derivative of the exponential profile at $x = x_k$ under continuous condition, and $n'(x_k) = -ab\exp(-ax_k)$.

Because the surface index n_c is unknown, another known condition should be added to solve the above linear equations. This added condition is that the point (n_0, x_0) lies in the curve from $(n_c, 0)$ to (n_1, x_1), which can be depicted as

$$n_0 = M_0\frac{(x_1 - x_0)^3}{6h_0} + M_1\frac{x_0^3}{6h_0} + \Big(n_c - \frac{M_0 h_0^2}{6}\Big)\frac{x_1 - x_0}{h_0} + \Big(n_1 - \frac{M_1 h_0^2}{6}\Big)\frac{x_0}{h_0}. \tag{6}$$

Finally, for an arbitrarily given series of $\{x_i\}$, the index distribution can be fitted with a very smooth profile according to the continuous first and second derivatives of all cubic spline functions, even at the interpolation points $[(n_1, x_1), (n_2, x_2), \cdots, (n_{k-1}, x_{k-1})]$. Then an exact analytic transfer matrix (ATM) method is introduced to solve the waveguide with this fitting index profile to obtain the corresponding effective indices n_i^{cal} $(i = 0, 1, \cdots, k)$. The dispersion equations of ATM method is presented as follows:[9]

$$\int_0^{x_t} k(x)dx + \Phi(\Gamma) = m\pi + \tan^{-1}\Big(\frac{p_0}{k_1}\Big) + \tan^{-1}\Big(\frac{p_t}{k_l}\Big), \quad (m, l = 0, 1, 2, \cdots),$$

where x_t is turning point of the monotonically decreasing index profile for the mode; $k(x) = [k_0^2 n^2(x) - \beta^2]^{1/2}$, $k_0 = 2\pi/\lambda$, and λ is the wavelength in air; β is the propagation constant, and $\beta = k_0 n(x_t) \cdot p = (\beta^2 - k_0^2 n_a^2)^{1/2}$, n_a is the refractive index of the cover layer. $k_1 = [k_0^2 n^2(0) - \beta^2]^{1/2}$, $n(0)$ is the surface index of the waveguide, $P_t = (\beta^2 - k_0^2 n_{eq}^2)^{1/2}$, n_{eq} is characterized as equivalent refractive index beyond x_t; $k_l \to 0$ as $l \to \infty$, m is the mode number, $\Phi(\Gamma)$ is interpreted as the phase contribution of the subwaves.

After solving the recovered interpolated index distribution of the waveguide, a new series of effective index $\{n_i^{\text{cal}}\}$ is obtained. Simultaneously, a new series of $\{x_i\}$ is acquired. We define and calculate the departure of the effective indexes between calculated values and exact values as $\Delta = \sum_{i=0}^{k}(n_i^{\text{cal}} - n_i)^2$. If it is still large enough, we substitute the new series of $\{x_i\}$ to Eq. (1), and a new index profile can be fitted with interpolation on the new series of points. Series effective indexes $\{n_i^{\text{cal}}\}$ and $\{x_i\}$ can be determined by the solution of the new index distribution in the waveguide with ATM method. Repeating the above approach the deviation Δ will become smaller. If this iteration process is convergent, the profile is approaching to the real profile. When Δ is close to zero (the typical value of Δ is 10^{-9} in our simulation), the refractive index profile is finally acquired.

It should be noted that for the facility of the iteration procedure, the first series of $\{x_i\}$ should be chosen so that the waveguide of the fitting profile has enough guiding modes. We can choose $\{x_i\}$ as arithmetic series and the interval is chosen to be $5\lambda - 8\lambda$, where λ is the wavelength in air.

In the above discussion we have noted that $n'(0)$, the first derivative of $n(x)$ at the surface $x = 0$, is given and may be altered. For a given $n'(0)$, the obtained profile may be impractical. Because the second derivative can reflect the concave and convex character of the curve, practicability of the index distribution can be judged by investigation on the series of second derivatives M_i $(i = 0, 1, \cdots, k)$. The judge rule is that every sign of $\{M_i\}$ before the first positive sign should be negative and every sign of $\{M_i\}$ after the first positive sign should be positive, or all of signs of $\{M_i\}$ are positive. As we know, for most of graded index profiles such as Gaussian profile, error function profile, Fermi profile and exponential profile, $n'(0)$ is less than or equal to zero. First, we set $n'(0) = 0$, and under this value we can obtain a convergent profile. Then, if the signs of $\{M_i\}$ in the profile satisfy the judge rule, the practical index profile is acquired, and if the signs of $\{M_i\}$ do not satisfy the judge rule, $n'(0)$ should be decreased until $\{M_i\}$ from the calculated profile according to the judge rule. It is proven by simulation results that from a series of effective indexes $\{n_i\}$, the index distribution can be precisely recovered very close to its exact profile with the iteration approach and the judge rule.

In order to investigate the reliability of this method, we give some typical examples of graded index profiles such as Gaussian, error function, Fermi and exponential profile. The cover layer is the air with index $n_a = 1.0$ and $\Delta = 10^{-9}$. All numerical simulations are performed with wavelength of 632.8 nm and the unit of length in micrometer. Every index profile in waveguides is also calculated under different mode numbers to verify the universality of the current method.

Firstly we consider waveguides with index distribution of Gaussian profile, which is described as $n(x) = 1.5 + 0.025\exp(-x^2/D_{\text{gauss}})$, D_{gauss} varies from 3 μm to 7 μm, allowing the waveguides to support 3–7 modes. All waveguides have uniform surface index with the value of 1.525. The numerical results are demonstrated in Fig. 1(a). A good superposition can be observed between the exact and calculated profiles in the guiding region. Some discrepancy exists in the retrieved profiles after the last mode because an exponential profile is assumed when $x \geq x_k$ in our calculations. The next example is implemented in waveguides with error function profile, which is depicted as $n(x) = 2.2 + \frac{n_{\text{eff}}}{2}\left[\text{erf}\left(\frac{1.0+x}{3.6}\right) + \text{erf}\left(\frac{1.0-x}{3.6}\right)\right]\Big/\text{erf}\left(\frac{1.0}{3.6}\right)$, where $n_{\text{erf}} = 0.01, 0.02, 0.035, 0.045$, corresponding to surface index, can stimulate 3, 4, 5 and 6 modes, respectively. The results are shown in Fig. 1(b). The recovered values of n_{erf} for each waveguide are 0.00998, 0.02012, 0.03516, 0.04518, which agree well with the exact values. Similar results are also obtained with exponential profile $n(x) = 2.47 + 0.01\exp(-x/D_{\text{exp}})$ and slowly varying Fermi profile $n(x) = 1.735 + n_{\text{femi}}/\{1 + \exp[(x-2.5)/0.7]\}$, as shown in Figs. 1(c) and 1(d).

To further verify the reliability of our method, we consider waveguides with only two guiding modes. Because of the invalidation of the judge rule on only 2 derivatives of the second order we just predict Gaussian, error function and slowly changing Fermi profiles except exponential profile, which can uniformly be calculated with $n'(0) = 0$. The effective indexes of TE_0 and TE_1 of Gaussian profile $n(x) = 1.5 + 0.018\exp(-x^2/2.8^2)$ are 1.5108 and 1.5033, respectively. The calculated result is shown in Fig. 2(a), which also agrees well with the exact profile. Figure 2(b) is achieved from error function profile $n(x) = 1.5 + 0.018 \times 1/2 \times \{\text{erf}[(1.0+x)/2.3] + \text{erf}[(1.0-x)/2.3]\}\text{erf}(1.0/2.3)$, which supports two guiding modes with effective indexes of 1.5099 and 1.5020. The obtained surface index is 1.51792, very close to the exact value. Although the obtained index distribution of Fermi profile $n(x) = 1.5 + 0.018/\{1 + \exp[(x-2.5)/0.7]\}$ expects a less accurate achievement, the result still gives a good information on the index amplitude and depth of the waveguide, as shown in Fig. 2(c).

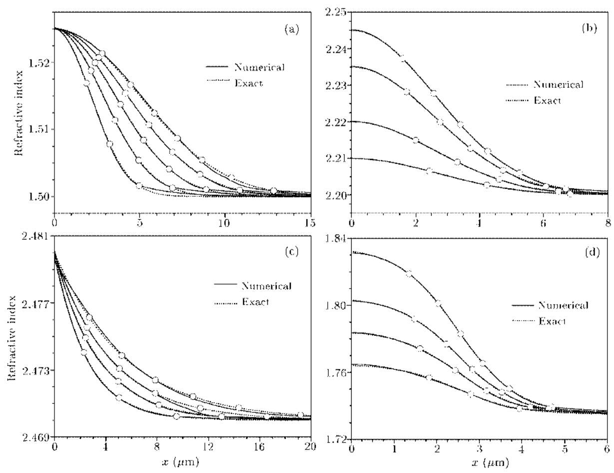

Fig. 1. (a) Recovery of Gaussian distribution by cubic spline function. D_{gauss} varies from 7 μm to 3 μm, allowing the waveguides to support 7–3 modes (from top to bottom). (b) Recovery of error function distribution by cubic spline function. Here $n_{\text{erf}} = 0.045, 0.035, 0.02, 0.01$ stimulate 6 modes, 5 modes, 4 modes, 3 modes, respectively (from top to bottom). (c) Recovery of exponential distribution by cubic spline function. The waveguides with $D_{\text{exp}} = 5.5, 4.3, 3.4, 2.5\,\mu$m support 6, 5, 4, 3 guiding modes, respectively (from top to bottom). (d) Recovery of slowly varying Fermi distribution by cubic spline function. Here $n_{\text{femi}} = 0.1, 0.07, 0.05, 0.03$ correspond with 6, 5, 4, 3 stimulated guiding modes, respectively (from top to bottom).

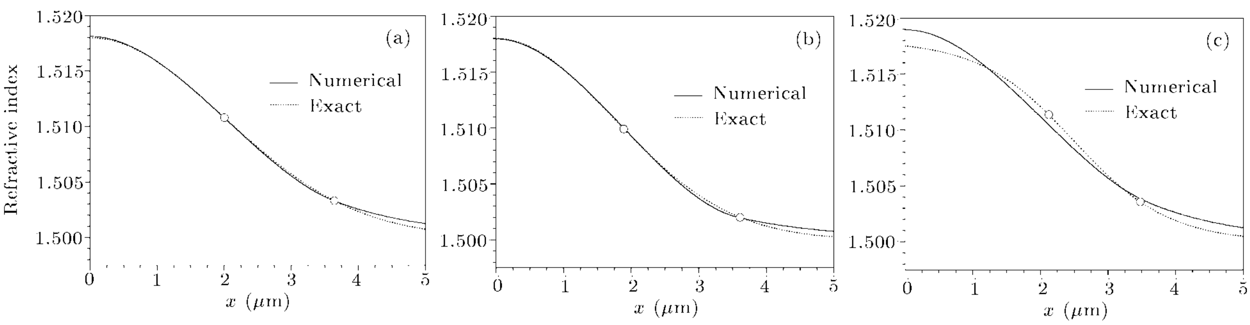

Fig. 2. Recovery of two-mode waveguide with (a) Gaussian profile, (b) error function profile, (c) slowly varying Fermi profile.

As the experimental values are usually measured in inevitable errors, it is essential to study the sensitivity of the current method to the experimental errors. Generally, errors of $0.5\%\delta n$, $1\%\delta n$ and $2\%\delta n$ are artificially added to the indices of a Gaussian profile $n(x) = 1.5 + \delta n \exp(-x^2/4^2)$, and δn is set to 0.25 for investigation. As shown in Fig. 3, an oscillatory character exhibits in the recovered profiles when the error increases, it is shown that this method can stand a typical experimental error of 1×10^{-4}.

In comparison with the commonly used IWKB method,[3] we consider a Gaussian profile $n(x) = 1.5 + 0.025 \exp(-x^2/5^2)$ supporting 5 guiding modes. The effective indexes are 1.51995, 1.51375, 1.50843, 1.50411, and 1.50107, respectively. Recovered profiles by the IWKB and the current method are shown in Fig. 4. It can be found that the current method can estimate the index profile in more accuracy than the IWKB method, especially in the recovery of surface index.

Finally, an experimentally fabricated x-cut $MgO:LiNbO_3$ waveguide diffused with the 80 nm ZnO film at 1000°C for 1 h is introduced for investigation.[11] The waveguide can support four guiding modes with effective indices of 2.2039, 2.2025, 2.2013, and 2.2006 at the wavelength of 632.8 nm. Figure 5 gives the recovered profile from the measured indices by our method and the predicted profile from the Ref. [11]. It can be found from the graph that the predicted profile from the reference is of much departure from the measured mode indices. We believe the recovered profile through all experimental data by the current method is in more creditable as well as the experimental data are measured in a good accuracy.

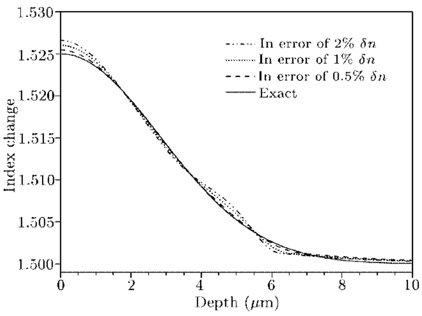

Fig. 3. Recovered Gaussian profiles $n(x) = 1.5 + \delta n \exp(-x^2/4^2)$, $\delta n = 0.25$ with errors of $0.5\%\delta n$, $1\%\delta n$ and $2\%\delta n$ added to odd-order mode indices.

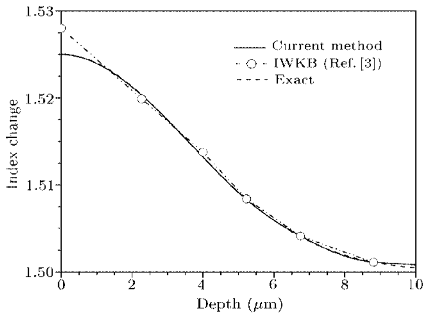

Fig. 4. Comparison of recovered profiles from effective indexes by the IWKB and the current method.

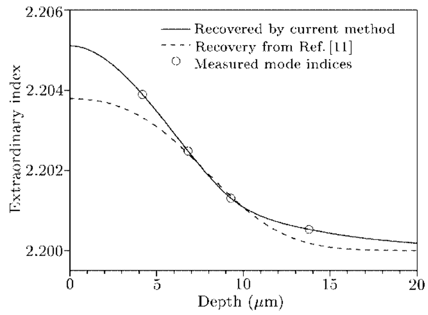

Fig. 5. Comparison of recovered extraordinary index profiles of an x-cut $MgO:LiNbO_3$ waveguide diffused with 80 nm ZnO film at 1000°C for 1 h.

In summary, we have demonstrated that the refractive index profile of graded index waveguide can be smoothly recovered in good accuracy with cubic spline interpolation functions based on the exact ATM method and simply iterative approaches. This method can predict the index profile of multimode waveguides with more modes and fewer modes from the effective indices. The explicit analysis provides a reliable and convenient technique in the approach of graded index profiling.

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